

# Quantitative multi-period reverse stress testing using quantum and simulated annealing (application to XVA)

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# Important Note

- Assad Bouayoun is XVA quantitative consultant at HSBC and prior to this was at Scotiabank™ in London.
- I am grateful to Andy Mason, Sheir Yarkoni, Andrew Green, Ryan Fergusson and Karin Bergeron for their help and continuous interaction.
- The views expressed in this presentation are the personal views of the speaker and do not necessarily reflect those of their company.
- Chatham House Rules apply to the reporting on this presentation and the comments of the speakers.

# Objectives

- Learn what is reverse stress testing and why it is important for financial risk management
- Understanding how to apply it to XVA / total valuation.
- Following a concrete example of optimisation using simulated and quantum annealing

# BIO

- Assad Bouayoun has over 15 years of quantitative analysis experience in investment banking. He is a quantitative finance specialist focusing on total valuation including funding and capital cost, xVA, risk, stress and reverse stress testing. He was responsible for designing industry standard hedging and pricing systems in equity derivatives during his time in Commerzbank, and had the same responsibility in credit derivatives while working for Credit Agricole, and also in xVA in institutions like Lloyds, RBS and Scotiabank. Currently he is leading the modelling team responsible for the research and development of the simulation engines used for exposure computation within HSBC in London.
- During the different projects Assad has undertaken in quantitative finance, he integrated new technologies such as Cloud, GPU and QPU; new design (parallelization using graphs) as well as new numerical methods such as AAD. He also participated to the firmwide data standardization and integration that are essential aspects of the success of these projects. He is also leading an effort to leverage quantum annealing for financial quantitative reverse stress testing.
- He holds a MSc in Mathematical Trading and Finance from Cass Business School and an MSc in Applied Mathematics and Computer Science from UTC ( University of Technology of Compiègne, France).

# Introduction

- Financial stress testing is becoming a dominant part of the arsenal built by regulators to protect the economic stability from the eventual distress of one or a series of financial institutions.
- To avoid a combinatorial explosion, a number of arbitrary choices are usually made in relation to the level of each shock, their combination and the time horizon. These assumptions although necessary, are limiting the effectiveness of this technique.
- This presentation investigates the possibility of inferring the worst case combination of scenarios maximising the XVA loss at a particular time horizon using quantum annealing developed by D-wave (<https://www.dwavesys.com/software>).
- Is it possible to benefit from the speed and power of QA by expressing the XVA reverse stressing as a QUBO (quadratic unconstrained binary optimisation) problem. As of today few attempts to exploit quantum annealing in finance have been successful. One of the reason was the difficulty to find and formulate a problem in a shape that could be solved particularly well with this technique and not or less with others.
- The model is implemented using D-wave API and compared to a simulated annealing benchmark.

# Structure

- **Definitions**
  - Qualitative reverse stress testing
  - Quantitative reverse stress testing
  - Application to XVA / Total valuation
- **Modelling**
  - XVA reverse stress testing formula
  - XVA reverse stress testing as a QUBO problem
  - Generalisation to multi-period case
- **Optimisation**
  - Application to a simple portfolio of swaps
  - Simulated annealing
  - Quantum annealing

# Definition

- The US, UK and European regulatory stress testing frameworks are designed to prevent failure of a systemic importance. This diverse set of scenarios is meant to uncover the worst possible loss for each relevant institution at a time horizon.
- Stress testing is finding the worst loss for a given set of shocks. It is not only a regulatory requirement but also a different way of observing the risks taken by a financial institution.
- Reverse stress testing is finding the set of shocks giving the worst loss.
- There will be always a worst case scenario, but its severity can be reduced considerably by finding the right mitigation.
- A more quantitative and systematic approach with less assumptions can not only help the regulator preventing another banking crisis, but also help banks better stress manage themselves.

# Definition **Qualitative reverse stress testing**

- The qualitative reverse stress testing is a top down analysis performed to reveal hidden vulnerabilities of the business.
- It usually starts from identifying the different blocks of risk and their dependences. Then adverse impact of different events are considered.
- It is a necessary step to build the worst case set of scenarios.
- Examples:
  - Failure of a parent company
  - Impact of reputational risk
  - Drying up of liquidity for an asset class
  - default of an important counterparty / country

**→ it is necessary but not sufficient.**



# Definition **Quantitative reverse stress testing**

- The idea is to lay the foundation of a more quantitative and systematic approach to reverse stress testing with less assumptions.
- We aim at improving the capital and funding management of a financial institution.
- One way of achieving this is to avoid the set of shocks maximising the loss due to unhedged risks (market, funding and credit).
- Every risk factor is shocked according to its dependencies and its historical variance.

# Definition Application to XVA / Total valuation

- We rely on the assumption that loss due to XVA is the right metric because:
  - Plain credit and market risk tend to be hedged
  - Residual risks contingent to counterparty and funding risk are more difficult to hedge and therefore subject to reserves
- The computation of XVA includes all the mitigants put in place by the bank:
  - Agreements on netting, collateral and close out, margin, break clause.
  - Aggregation respecting netting, funding and capital sets
- We can define  $XVA = CVA + DVA + COLVA + FCA + MVA + KVA$  ( or something else)
- Or we can start by a simpler definition  $XVA = CVA + FVA$

# Definition Incomplete introduction to XVA

## Deriving CVA in a simple case by semi-replication

If we consider a set of contracts defined by an aggregated payoff function  $\Phi(X_T)$  depending on the financial market  $\Psi_t$  defined above, the arbitrage free value at time  $t \leq T$  is therefore the expected discounted value under probability measure  $\mathbb{Q}$  using the risk free rate  $r(t)$  of the payoff  $V(X_t, t)$ :

$$V(x, t) = \mathbf{E} \left[ e^{\int_t^T r(s) ds} \Phi(X_T) | X_t = x \right] \quad (9)$$

The Feynman-Kac theorem links SDE (Stochastic Differential Equations) with PDE (Partial Differential Equations) by expressing the solution of the partial differential equation as an expected value of a function of the stochastic differential equation:

$$\frac{\partial V(x, t)}{\partial t} dt + \sum_{i=1}^N \mu_i \frac{\partial V(x, t)}{\partial x_i} dx_i + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \sigma_i \sigma_j \frac{\partial^2 V(x, t)}{\partial x_i \partial x_j} dx_i dx_j = 0 \quad (10)$$

The proof consists in starting from the solution, finding the martingale by taking its differential and then set the deterministic part to zero.

A special case of this theorem establish the relation between GBM (Geometric Brownian Motion) and BS (Black-Scholes).

# Definition **Incomplete introduction to XVA**

## Deriving CVA in a simple case by semi-replication

We start from the same European payoff  $\Phi$  on an underlying which market price is modeled as a stochastic process  $X(t)$  following a geometric Brownian motion. The underlying pays a yield  $y_x$  and can be repoed (overnight repurchase agreement) at rate  $q_x$ . This derivative is bought from a counterparty  $C$  by the entity  $B$ .

A collateral agreement is signed with collateral amount  $CA_t$  and close out condition  $g_c(CO_c, CA_t)$  on counterparty  $C$  default depending on  $CO_c$ , the close out value of the derivative.

If we consider  $V'_t$  and  $V_t$ , the risk neutral values of the derivative with and without counterparty credit risk and  $V_t$ , the sum of value adjustments is then:

$$V(t) - V'(t) = xVA(t) \quad (12)$$

There are only three financial risks affecting the value of this derivative for  $B$ :

- the change in the underlying price
- the possibility of a default of the counterparty
- the change in the interest rate considered as risk free (OIS)

# Definition

## Incomplete introduction to XVA

### Deriving CVA in a simple case by semi-replication

The value of our derivative is the cost of its replication portfolio. The replication portfolio  $\Pi_t$  is a basket of primary market instruments built in such a manner that a change in its value is offset by the change in value of the derivative. It includes all the necessary risk mitigants, the underlying, the bonds issued by the counterparties, the collateral and the cash amounts accumulated by borrowing or lending them:

- the underlying at a rate  $\beta_x = y_x - q_x$  such that  $d\beta_x = \delta(y_x - q_x)X_t dt$
- the collateral at a rate  $\beta_{ca}$  such that  $d\beta_{ca} = -r_{ca}CA_t dt$
- the bond  $P_C$  at a repo rate  $d\beta_c = -q_c P_t^c dt$  and zero recovery and zero coupon with jump to default  $J_c$  such that  $dP_t^c = r^c P_t^c dt - P_t^c dJ^c$
- the risk free bond  $P$  at a risk free rate  $r$  such that  $dP_t = rP_t dt$

The portfolio can then be written as follows:

$$\Pi_t = \delta X_t + \alpha_c P_t^c + \alpha P_t - CA_t + \beta_{ca} + \beta_x \quad (13)$$

# Definition Incomplete introduction to XVA

## Deriving CVA in a simple case by semi-replication

The infinitesimal change in replication portfolio is then:

$$d\Pi_t = \delta dX_t + \alpha_c dP_t^C + \alpha dP_t - CA_t + d\beta_{ca} + d\beta_x \quad (14)$$

Replacing each term by its expression gives :

$$d\Pi_t = \delta dX_t + \delta(y_x - q_x)X_t dt + \alpha r P_t dt + r^c P_t^c dt - P_t^c dJ^c + \alpha_c q_c P_t^c dt - r_{ca} CA_t dt \quad (15)$$

The partial differential equation (PDE) of the counterparty risk risk adjusted value (with  $\Delta V_t' = g_c(CO_c, CA_t) - V_t'$  defined as the change in value of the derivative at time of default) can be derived easily by applying Ito lemma:

$$dV_t' = \frac{\partial V_t'}{\partial t} dt + \frac{\partial V_t'}{\partial X} dX + \frac{1}{2} \sigma(t) X^2 \frac{\partial^2 V_t'}{\partial X^2} + \Delta V_t' dJ^c \quad (16)$$

The standard close out function  $g_c(CO_c, CA_t)$  is often defined as a function of the recovery rate  $R^c$  and the value of the derivative  $V_t$ :

$$g_c(CO_c, CA_t) = R^c (V_t - CA_t)^+ + (V_t - CA_t)^- + CA_t \quad (17)$$

# Definition Incomplete introduction to XVA

## Deriving CVA in a simple case by semi-replication

The infinitesimal evolution of the risk less portfolio  $\Pi_t + V_t'$  value constituted of the derivative and its replication portfolio must be on average null, or at least driftless:

$$d\Pi_t + dV_t' = A dt + B dX + C dJ^c \quad (18)$$

with

$$\begin{cases} A = \left[ \frac{\partial V_t'}{\partial t} dt + \frac{1}{2} \sigma(t) X^2 \frac{\partial^2 V_t'}{\partial X^2} + \delta(y_x - q_x) X_t + \alpha r P_t + \alpha_c (r_{ca} - q_c) P_t^c - r_{ca} C A_t \right] \\ B = \left[ \delta + \frac{\partial V_t'}{\partial X} \right] \\ C = \Delta V_t' - \alpha_c P_t^c \end{cases} \quad (19)$$

# Definition Incomplete introduction to XVA

## Deriving CVA in a simple case by semi-replication

The variation of the value of this portfolio is null if each term  $A$ ,  $B$  and  $C$  are null.  
This implies:

$$\begin{cases} \frac{\partial V'_t}{\partial t} dt + \frac{1}{2} \sigma(t) X^2 \frac{\partial^2 V'_t}{\partial X^2} + \delta(y_x - q_x) X_t + \alpha r P_t + \alpha_c (r_{ca} - q_c) P_t^c - r_{ca} C A_t = 0 \\ \delta = -\frac{\partial V'_t}{\partial X} \\ \alpha_c P_t^c = g_c(CO_c, C A_t) - V'_t \end{cases} \quad (20)$$

We can then replace  $V'(t)$  by its value ( $V'(t) = V(t) - xVA(t)$ ), then eliminate  $V(t)$  because it is solution of Black, Scholes and Merton PDE:

$$dV_t = \frac{\partial V_t}{\partial t} dt + \delta(y_x - q_x) \frac{\partial V_t}{\partial X} dX + \frac{1}{2} \sigma(t) X^2 \frac{\partial^2 V_t}{\partial X^2} + -rV = 0 \quad (21)$$



# Definition Incomplete introduction to XVA

## Deriving CVA in a simple case by semi-replication

We assume the difference between repo rate and yield on the counterparty bond is corresponding to the hazard rate of the counterparty default process  $\lambda_c = r_c - q_c$ , and define the collateral spread of the risk free rate as  $s_{ca} = r_{ca} - r$ . Here  $\alpha$  is the quantity of risk free bond used to finance the collateral and the derivative defined in the financing formula  $V_t' - CA_t + \alpha P_t = 0$ . We can rewrite the PDE for  $V_t'$  as follows:

$$\begin{cases} \frac{\partial V_t'}{\partial t} dt + \frac{1}{2} \sigma(t) X^2 \frac{\partial^2 V_t'}{\partial X^2} - (y_x - q_x) \frac{\partial V_t'}{\partial X} + \alpha r P_t + \lambda_c (g_c(CO_c, CA_t) - V_t') - (s_{ca} - r) CA_t = 0 \\ \frac{\partial V_t'}{\partial t} dt + \frac{1}{2} \sigma(t) X^2 \frac{\partial^2 V_t'}{\partial X^2} - (y_x - q_x) \frac{\partial V_t'}{\partial X} - r V_t' + \lambda_c (g_c(CO_c, CA_t) - V_t') - s_{ca} CA_t = 0 \end{cases} \quad (22)$$

Removing the PDE for  $V_t$  we obtain the PDE for  $xVA_t$ :

$$\begin{aligned} \frac{\partial xVA_t}{\partial t} dt - (y_x - q_x) \frac{\partial xVA_t}{\partial X} dX + \frac{1}{2} \sigma(t) X^2 \frac{\partial^2 xVA_t}{\partial X^2} - (r + \lambda_c) xVA_t = \\ - \lambda_c (g_c(CO_c, CA_t) - V_t) + s_{ca} CA_t \end{aligned} \quad (23)$$

# Definition **Incomplete introduction to XVA**

## **Introduction to FVA by relaxation of risk free rate self financing**

Risk free self financing must be relaxed as it is not anymore a realistic assumption in modern finance. The financing formula must rely on risky financing and allow for hedging of defaults. We therefore need a new degree of freedom. As we remove the possibility of funding using the risk free bond, we must introduce two new bonds to finance our new risk free portfolio.

For this we need an extra degree of freedom. We then introduce the bonds  $P^{B1}$  and  $P^{B2}$  issued by the bank  $B$  with repo rate  $q_{b1} < q_{b2}$  ( here the first bond is more senior than the second bond) and recoveries  $R^{B1}$  and  $R^{B2}$ :

$$\begin{cases} dP^{B1} P^{B1-1} = r^{B1} dt - (1 - R^{B1}) dJ^B \\ dP^{B2} P^{B2-1} = r^{B2} dt - (1 - R^{B2}) dJ^B \\ r^{B1} - r = (1 - R^{B1}) \lambda_B \\ r^{B2} - r = (1 - R^{B2}) \lambda_B \end{cases} \quad (32)$$

# Definition Incomplete introduction to XVA

## Introduction to FVA by relaxation of risk free rate self financing

The infinitesimal variation of the risk free portfolio can be written as follows:

$$\begin{aligned} d\Pi_t = & \delta dX_t + \delta(y_x - q_x)X_t dt + r^c P_t^c dt - P_t^c dJ^c + \\ & \alpha_{b1} dP_t^{B1} + \alpha_{b2} dP_t^{B2} - \alpha_{B1} q_{B1} P_{B1} dt - \alpha_{B2} q_{B2} P_{B2} dt \\ & \alpha_c q_c P_t^c dt - r_c C A_t dt \end{aligned} \quad (36)$$

We have then

$$d\Pi_t + dV_t' = A dt + B dX + C dJ^c + D dJ^b \quad (37)$$

with

$$\left\{ \begin{aligned} A = & \frac{\partial V_t'}{\partial t} dt + \frac{1}{2} \sigma(t) X^2 \frac{\partial^2 V_t'}{\partial X^2} + \delta(y_x - q_x) X_t - r_c C A_t + \\ & \alpha_c (r_c - q_c) P_t^c + \alpha_{B1} (r_{B1} - q_{B1}) P_t^{B1} + \alpha_{B2} (r_{B2} - q_{B2}) P_t^{B2} \\ B = & \delta + \frac{\partial V_t'}{\partial X} \\ C = & \Delta V_t'^c - \alpha_c P_t^c \\ D = & \Delta V_t'^b - \alpha_{B1} (1 - R^{B1}) P_t^{B1} - \alpha_{B2} (1 - R^{B2}) P_t^{B2} \end{aligned} \right. \quad (38)$$

# Definition **Incomplete introduction to XVA**

## Introduction to FVA by relaxation of risk free rate self financing

Its variation is null if each term is null. The last term corresponding to the hedge of the default of the bank (self) is problematic because it is difficult to rebalance the portfolio around the default event. We assume the bank is not using repo with its own bond and does not hedge its own default in case of gain to protect the debtholder. An hedging error  $\epsilon_h = D$  is therefore introduced:

$$\alpha_{B1}r^{B1}P_t^{B1} + \alpha_{B2}r^{B2}P_t^{B2} = r_cCA_t - (r + \lambda_b)V_t' - \lambda_b(\epsilon_h - g_b(CO_b, CA_t)) \quad (39)$$

We use the formula canceling  $B$ ,  $C$ , the financing formula, the default intensity formulas to rearrange the equation for  $A$  and we obtain the PDE for the adjusted price with the boundary condition  $V_{T, X_T}' = H(X_T)$  :

$$\begin{aligned} \frac{\partial V_t'}{\partial t} dt + \frac{1}{2} \sigma(t) X^2 \frac{\partial^2 V_t'}{\partial X^2} - (y_x - q_x) X \frac{\partial V_t'}{\partial X} - (r + \lambda_b + \lambda_c) V_t' = \\ r_{ca} CA_t - \lambda_b g_b(CO_b, CA_t) - \lambda_c g_c(CO_c, CA_t) - \epsilon_h \lambda_b \end{aligned} \quad (40)$$

# Definition **Incomplete introduction to XVA**

## Introduction to FVA by relaxation of risk free rate self financing

Semi-replication with only one bond: The bank issues only one bond yielding  $r^b = r + s^b$ , with  $s^b$ , the funding spread. This bond is used to fund the derivative and its collateral and there is no other degree of freedom to hedge the bank default. The error becomes  $\epsilon_h = g_b(CO_b, CA_t) + P^B$  for:

$$\begin{cases} \alpha_{b1} P^{B1} = 0 \\ \alpha_{b2} P^{B2} = V_t + XVA_t \end{cases} \quad (46)$$

Using exactly the same logic as before we get the following formula for  $XVA_t$ :

$$\begin{aligned} \frac{\partial XVA_t}{\partial t} dt - (y_x - q_x) X \frac{\partial XVA_t}{\partial X} dX + \frac{1}{2} \sigma(t) X^2 \frac{\partial^2 XVA_t}{\partial X^2} - (r^b + \lambda_c) XVA_t = \\ r_{ca} CA_t - \lambda_c (g_c(CO_c, CA_t) - V_t) - s^b (g_b(CO_b, CA_t) - V_t) \end{aligned} \quad (47)$$

# Definition **Incomplete introduction to XVA**

## Introduction to FVA by relaxation of risk free rate self financing

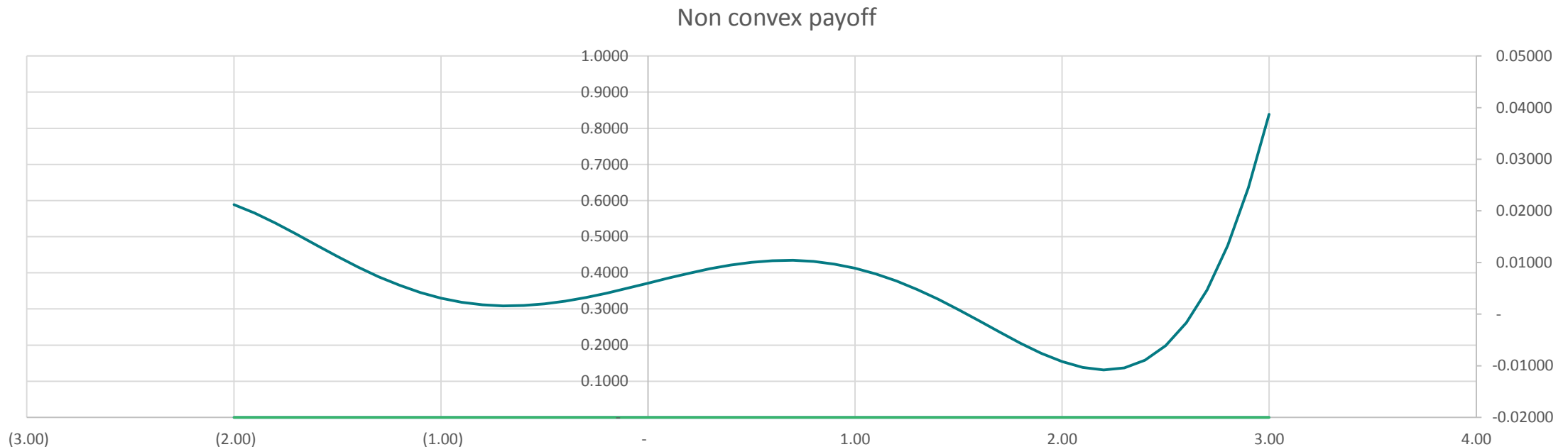
We have directly:

$$\begin{cases} CVA_t = -(1 - R_c)\lambda_c \int_t^T e^{-\int_t^s r^b(u) + \lambda_c(u) du} \mathbf{E}[V(s) - CA(s)]^+ ds \\ FVA_t = - \int_t^T s^b(s) e^{-\int_t^s r^b(u) + \lambda_c(u) du} \mathbf{E}[V(s) - CA(s)] ds \end{cases} \quad (48)$$

We can observe the disappearance of the bank own hazard rate and the discounted at the funding rate. These formula are widely used currently.

# Optimisation Simulated annealing results

- If we include non convex payoffs, the XVA is more likely to become non convex:



- Non convexity can also come from hedged portfolio where the first and second order derivatives are often partially cancelled.
- Collateral and initial margin can also accentuate non convexity, create discontinuities.

# Modelling

- This section shows how the XVA reverse stress testing problem can be defined and how it can be reformulated into a QUBO problem.
- As part of the XVA risk management (see AG XVA ), we want to identify the worst case market scenarios at several given time horizons for a group of portfolios of financial derivatives.
- A worst case market scenario can be defined as the scenario maximizing the losses due to XVA variation. This loss can be computed as a function of its aggregated partial derivatives (known in finance as the sensitivities or Greeks)



# Modelling XVA reverse stress formula

is the sum of all value adjustment following the standard convention of negativity for costs and positivity for benefits.

is theta, the sensitivity of XVA with respect to time. It can be computed analytically.

are the gamma or cross gamma sensitivities with respect to the factors. They are computed in the same way as delta.

$$XVA(t+dt, \vec{x}+\vec{dx}) = XVA + \frac{\partial XVA}{\partial t} dt + \sum_{i=1}^N \frac{\partial XVA}{\partial x_i} dx_i + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \frac{\partial^2 XVA}{\partial x_i \partial x_j} dx_i dx_j \quad (1)$$

is the vector corresponding to all the credit market and correlation factors influencing XVA. This vector can have different representation. It is possible to switch from one representation into another by Jacobian transformation.

are the delta or Vega sensitivities with respect to the factors. They usually are computed analytically using finite difference or AAD.

# Modelling XVA reverse stress formula

- Choosing the vector of risk factors is not a trivial exercise as factors are linked by no-arbitrage rules.
- A scenario where each risk factor is shocked independently won't therefore be realistic.
- A reasonable remediation would be an historical principal component analysis to select the most important independent factors and their share of the variability of the original data.
- It is then easy to change variable in equation 1 from the original set of factors to the principal factors.
- An welcomed consequence is a reduction of the dimensionality of the problem.
- This solution would then use historical data to determine the set of independent factors and the size of the shocks.

# Modelling XVA reverse stress formula

- If we assume that XVA is regular enough over one day, we can inject daily shocks to all the factors in the Taylor approximation to compute the stressed XVA:

$$XVA(t+\Delta t, \vec{x}+\vec{\Delta x}) \approx XVA + \frac{\partial XVA}{\partial t} \Delta t + \sum_{i=1}^N \frac{\partial XVA}{\partial x_i} \Delta x_i + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \frac{\partial^2 XVA}{\partial x_i \partial x_j} \Delta x_i \Delta x_j \quad (2)$$

- These factors are then employed in the calibration of the stochastic risk factors: interest rate and foreign exchange rate principally (but this could extend to inflation credit and equity as well). The model can produce the XVA and sensitivities of each value adjustment with respect to each factor. This computation is expensive; it is therefore preferable to find a short optimisation path for the maximisation of the loss due to XVA.

# Modelling XVA reverse stress formula

- Traditionally, financial institutions rely on a small set of market scenarios chosen by practitioners in cooperation with regulators. This can be problematic as future stresses won't necessarily be assembled in the same manner as those driven by past experience.
- For XVA in particular, computing impacts for all combinations of scenarios would be too expensive. To circumvent these limitations, we investigate the use of several search algorithms.
- In this regard, quantum annealing seems to be particularly powerful for the resolution of quadratic unconstrained binary optimisation. For this the reverse stress testing problem defined precedently must be reformulated into a QUBO problem.

# Modelling QUBO formulation

- Quantum annealing is a technique used to solve quadratic unconstrained binary problems. It is minimising an energy function over a set of binary variables (C McGeoch, Adiabatic Quantum Computation and Quantum Annealing):

$$E(X_1, X_2, \dots, X_N) = \sum_{i=1}^N c_i X_i + \sum_{i=1}^N \sum_{j=1}^i Q_{ij} \times X_i \times X_j \quad (3)$$

With

$$\begin{cases} X_i \in \{0, 1\} \\ c_i, Q_{ij} \in R \end{cases} \quad (4)$$

# Modelling XVA reverse stress testing as a QUBO problem

Each shock  $\Delta x_i$ , once determined can be applied positively or negatively. If we define the absolute relative shock on  $x_i$  as  $\Delta x_i$ , then the binary variable  $X_i$  can take the value 0 if the shock is  $\Delta x_i$  or 1 if the shock is  $-\Delta x_i$ . We can write that the total shock for the factor  $i$  is  $(2X_i - 1)\Delta x_i$ .

As XVA is defined as a cost when negative, the loss is maximised when the XVA is minimised:

$$\begin{aligned} \operatorname{argmin}_{\vec{X}} \left[ XVA(t + \Delta t, \vec{x} + \Delta \vec{x}, \vec{X}) \right] &\approx XVA + \frac{\partial XVA}{\partial t} \Delta t + \\ &\sum_{i=1}^N \frac{\partial XVA}{\partial x_i} (2X_i - 1) \Delta x_i + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \frac{\partial^2 XVA}{\partial x_i \partial x_j} (2X_i - 1) \Delta x_i (2X_j - 1) \Delta x_j \end{aligned} \quad (5)$$

# Modelling XVA reverse stress testing as a QUBO problem

$$\operatorname{argmin}_{\vec{x}} \left[ \sum_{i=1}^N \frac{\partial XVA}{\partial x_i} X_i \Delta x_i + \sum_{i=1}^N \sum_{j=1}^N \frac{\partial^2 XVA}{\partial x_i \partial x_j} X_i \Delta x_i X_j \Delta x_j \right] \quad (6)$$

If we use the following notations:

The minimisation does not change as long as we include in the argmin function only factors that are independent from the argument of the minimisation. This equation is then equivalent to :

- $c_i \in R$  the linear coefficients /  $c_i = \frac{\partial XVA}{\partial x_i} \Delta x_i$
- $Q_{ij} \in R$  the quadratic coefficient /  $Q_{ij} = \frac{\partial^2 XVA}{\partial x_i \partial x_j} \Delta x_i \Delta x_j$
- $E(\vec{X}) \in R$  the energy to minimize /  
 $E(\vec{X}) = - \left[ XVA(t + \Delta t, \vec{x} + \vec{\Delta x}, \vec{X}) + XVA(t + \Delta t, \vec{x} + \vec{\Delta x}) - 2XVA - 2 \frac{\partial XVA}{\partial t} \Delta t \right]$

We finally obtain the QUBO formulation :

$$\operatorname{argmin}_{\vec{X}} \left[ E(\vec{X}) \right] = \sum_{i=1}^N c_i X_i + \sum_{i=1}^N \sum_{j=1}^N Q_{ij} X_i X_j \quad (7)$$

# Modelling

## Generalisation to multi-period case

- The XVA reverse stress testing can be generalised to multi-period case by running successively the XVA and sensitivity calculation on the worst case combination of scenario computed by quantum annealing for the precedent day.
- Direct reverse stress testing XVA over a longer period is not possible for several reasons.
  - First, the XVA function is not regular enough to be approximated by a Taylor expansion for large changes in its variables.
  - Second, the XVA computation can run in the future only if each trade is ageing correctly. It means the correct population of the different fixings and the application of all exercise conditions. This can be done only if the shocks have been identified for each day preceding the stress run. This issue can be solved by iterating through the computation of the worst case scenario and using it as the base case for the next day. This leads to the construction of the path to the worst loss for a long period that could extend to one month or even one year.



# Modelling

## Generalisation to multi-period case

- The algorithm can be summarized as follows:
  - 1 Analyse risk factors and determine independent factors and correlations shocks.
  - 2 Compute XVA and sensitivities for date  $t$ .
  - 3 Find the binary variable corresponding to the shocks maximising the XVA using quantum or simulated annealing.
  - 4 Age all the trades by applying all fixings and exercise, and break conditions.
  - 5 Compute the new XVA value for  $t=t+1$  using the precedent combination of shocks.
  - 6 Restart at step 2 until reaching the desired period (3M, 1Y)
- The solution is then a sequence of binary numbers corresponding to the combination of shocks for each date.

# Optimisation risk factors

- The numerical experiment is based on a simple implementation of a three factor model for interest rates and foreign exchange:
  - USD and EUR interest rates follow each a HW process
  - EURUSD foreign exchange rate follows a pseudo lognormal process
- There are 12 different risk factors:
  - USD short interest rate
  - USD volatility of the short interest rate
  - EUR short interest rate
  - EUR volatility of the short interest rate
  - EURUSD fx spot
  - EURUSD volatility of the fx spot
  - Correlation EUR / USD
  - Correlation EUR / EURUSD
  - Correlation USD / EURUSD
  - Funding spread
  - Credit spread
  - Recovery rate

# Optimisation Application to a simple portfolio of swaps

- We build randomly a portfolio of interest rate swaps with random:
  - Side,
  - maturity,
  - Payment,
  - Frequency

	Minimum	Initial value	Maximum	Bump (sensitivity)	Bump (shock)	
FUNDING	0.0001	0.0010	0.0200	0.0001	0.0050	
CREDIT	0.0001	0.0100	0.0200	0.0001	0.0050	
RECOVERY	0.0100	0.4000	1.0000	0.0100	0.5000	
IRDOM	0.0100	0.0500	0.2000	0.0100	0.5000	
IRFOR1	0.0100	0.0500	0.2000	0.0100	0.5000	
FX1	0.5000	1.1000	2.0000	0.0010	0.0500	
IRDOMVOL	-	-	0.0500	0.0010	0.0050	
IRFORVOL1	-	-	0.0500	0.0010	0.0050	
IRFXVOL1	-	-	0.0500	0.0010	0.0050	
IRIRCORR12	-	0.6000	0.4000	0.6000	0.0100	0.0500
IRDOMFXCORR1	-	0.6000	0.4000	0.6000	0.0100	0.0500
IRFORFXCORR1	-	0.6000	0.4000	0.6000	0.0100	0.0500

- We set the input parameters as displayed above.

# Optimisation results

- Simulated market values, CVA and FVA are computed for a particular portfolio.
- The corresponding Jacobian (first order derivatives) vector and the Hessian matrix (second derivatives) are also computed using AAD.
- First and second derivatives of each risk factors are multiplied by the corresponding shock. A test is performed to remain within the range defined by the precedent table.
- Running the model we obtain the vector and the matrix of the QUBO.

	Jacobian x Shock ci	Hessian x Shock i x Shock j Qij
FUNDING	- 0.47	0.00 68.97 0.08 - 228.41 109.67 0.52 - 3.89 1.10 - 0.19 - 0.27 - 0.20 - 0.21
CREDIT	- 57.19	68.97 0.00 - 69.29 - 282.01 35.79 - 68.62 - 70.67 - 68.34 - 69.34 - 69.37 - 69.34 - 69.35
RECOVERY	- 0.24	0.08 - 69.29 - 0.00 - 228.42 109.56 0.43 - 3.96 1.01 - 0.27 - 0.35 - 0.28 - 0.30
IRDOM	455.68	- 228.41 - 282.01 - 228.42 - 2.32 338.14 228.92 224.51 229.50 228.21 228.14 228.21 228.19
IRFOR	- 219.58	109.67 35.79 109.56 338.14 0.46 - 108.97 - 113.56 - 108.57 - 109.86 - 109.94 - 109.87 - 109.88
FX	- 1.43	0.52 - 68.62 0.43 228.92 - 108.97 0.00 - 4.41 0.58 - 0.71 - 0.78 - 0.71 - 0.73
IRDOMVOL	8.14	- 3.89 - 70.67 - 3.96 224.51 - 113.56 - 4.41 0.04 4.99 3.70 3.62 3.69 3.68
IRFORVOL	- 2.71	1.10 - 68.34 1.01 229.50 - 108.57 0.58 4.99 - 0.01 - 1.29 - 1.36 - 1.30 - 1.31
IRFXVOL	- 0.01	- 0.19 - 69.34 - 0.27 228.21 - 109.86 - 0.71 3.70 - 1.29 0.00 - 0.07 - 0.01 - 0.02
IRRCORR	0.12	- 0.27 - 69.37 - 0.35 228.14 - 109.94 - 0.78 3.62 - 1.36 - 0.07 0.00 0.07 0.05
IRDOMFXCORR	- 0.02	- 0.20 - 69.34 - 0.28 228.21 - 109.87 - 0.71 3.69 - 1.30 - 0.01 0.07 0.00 - 0.02
IRFORFXCORR	0.01	- 0.21 - 69.35 - 0.30 228.19 - 109.88 - 0.73 3.68 - 1.31 - 0.02 0.05 - 0.02 0.00

Figure 3: Figure displaying the QUBO vector and matrix

# Optimisation results

- The properties of the Hessian are depending on market conditions and on the portfolio. The Hessian used as example has positive and negative eigenvalues.
- The pricing function is an aggregation of composition of functions that are not always convex.
- The non convexity is in particular coming from the payoff functions, collateral and initial margin.
- The hyperrectangle defining the space all possible combination of inputs is bounded by financial constraints:
  - Volatilities can't be negative
  - Foreign exchanges can't be negative
- Some variables are hitting the boundary at the minimum because the function is convex to their respect. In that case they could be removed from the optimisation to reduce the dimensionality: here credit spread is monotonously increasing xva in that model.
- The system is not fully unconstrained:
  - Total variance of the foreign exchange must be null or positive.
  - Correlation matrix must be semi-definite and positive

# Optimisation results

- Exhaustive results : If we compute all the set of possible scenarios (  $2^{12} = 4096$ ), we obtain easily the minimum ( scenario number, energy and the binary vector solution):

Scenario	Energy	FUNDING	CREDIT	RECOVERY	IRDOM	IRFOR	FX	IRDOMVOL	IRFORVOL	IRFXVOL	IRIRCORR	IRDOMFXCORR	IRFORFXCORR
1	-	0	0	0	0	0	0	0	0	0	0	0	0
2	-0.02	0	0	0	0	0	0	0	0	0	0	0	1
3	-0.02	0	0	0	0	0	0	0	0	0	0	1	0
4	-0.05	0	0	0	0	0	0	0	0	0	0	1	1
5	0.13	0	0	0	0	0	0	0	0	0	1	0	0
6	0.26	0	0	0	0	0	0	0	0	0	1	0	1
7	0.26	0	0	0	0	0	0	0	0	0	1	1	0
8	0.38	0	0	0	0	0	0	0	0	0	1	1	1
9	-0.03	0	0	0	0	0	0	0	0	1	0	0	0
1277	-2,393.20	0	1	0	0	1	1	1	1	1	1	0	0
1278	-2,812.43	0	1	0	0	1	1	1	1	1	1	0	1
1279	-2,812.40	0	1	0	0	1	1	1	1	1	1	1	0
1280	-3,231.64	0	1	0	0	1	1	1	1	1	1	1	1
1281	-257.78	0	1	0	1	0	0	0	0	0	0	0	0
1282	-3.04	0	1	0	1	0	0	0	0	0	0	0	1
1283	-3.04	0	1	0	1	0	0	0	0	0	0	1	0
1284	251.69	0	1	0	1	0	0	0	0	0	0	1	1
4095	-61.99	1	1	1	1	1	1	1	1	1	1	1	0
4096	-25.64	1	1	1	1	1	1	1	1	1	1	1	1
Min = 1280	-3,231.64	0	1	0	0	1	1	1	1	1	1	1	1

Table 4: Table showing all the different risk factor scenarios and their energy

# Optimisation

## Simulated annealing applied to QUBO

Simulated annealing is an algorithm used for solving unconstrained optimization problems. The method models the physical process of heating a material and then slowly lowering the temperature to decrease defects, thus minimizing the system energy.

A binary version of this algorithm. is used to solve the QUBO system. This algorithm can be described as follows: the neighboring function returns a list of vectors where at most one binary variable has changed. For each neighboring vector, the energy is computed. One of them is selected randomly. If its energy is lower than the previous state, it is replaced. Else it is replaced with a probability depending on the difference of energy between the two states and the number of iterations of the algorithm. It means that the algorithm's acceptance of non optimal states is decreasing as it converges in time and energy.

As the number of iterations grows, the probability of selecting an optimal state converges to one. The optimal solution is found when there is no better neighbor and the probability of choosing the best is one.

# Optimisation Simulated annealing applied to QUBO

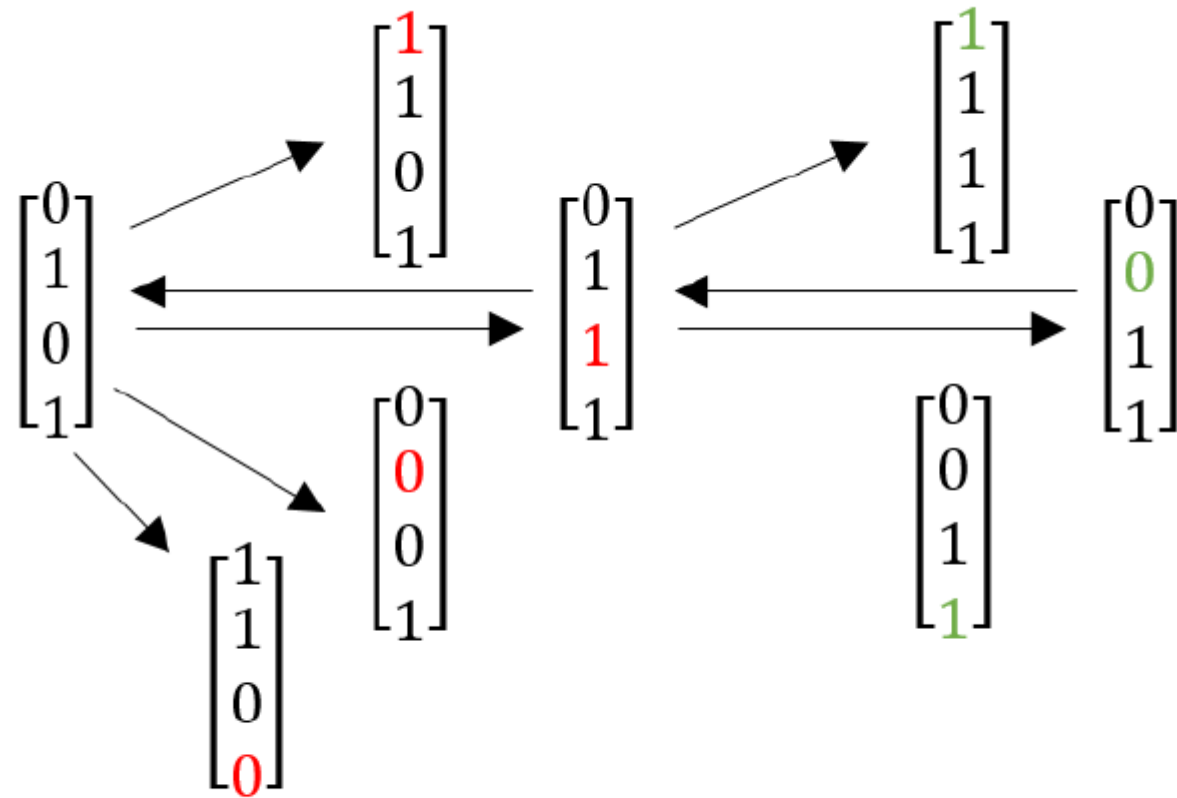


Figure 1: simulated annealing selection of solution process



# Optimisation Simulated annealing results

- The simulated annealing algorithm choose the best of three different mutations of the scenario with a probability increasing with iteration (time) to avoid local minimums at the start.

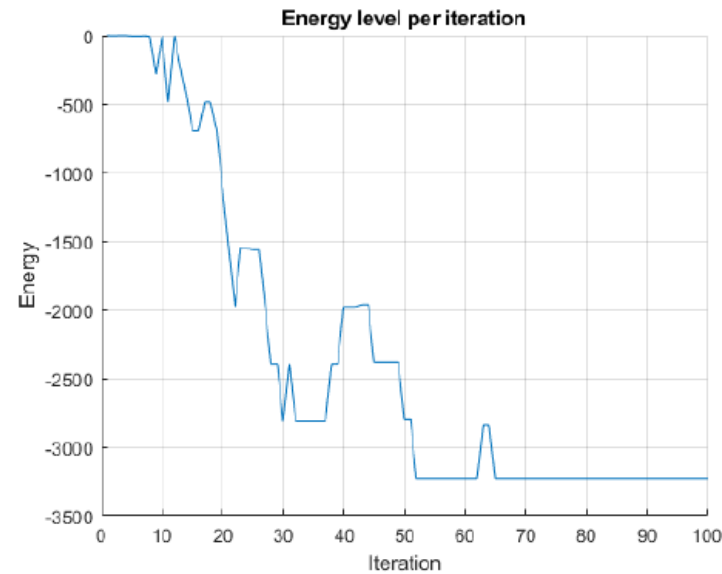
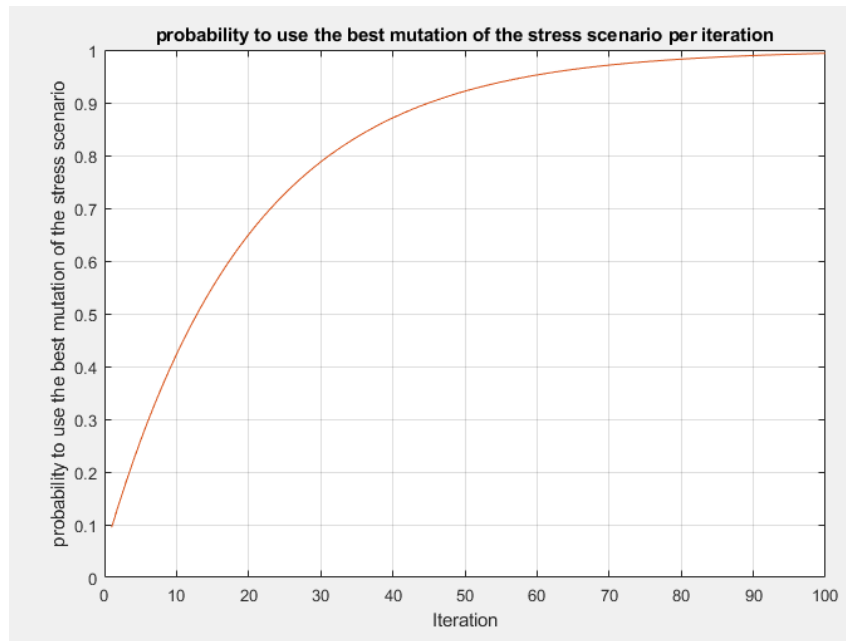
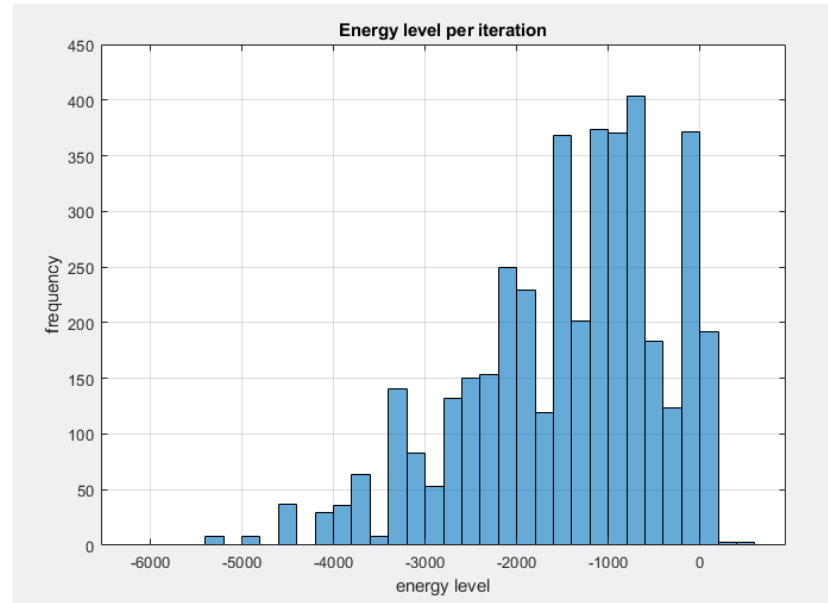


Figure 6: Figure showing the energy per iteration

Simulated annealing is converging in this case really fast as only 70 iterations are needed to reach the minimum energy. It is however difficult to extrapolate.

# Optimisation Simulated annealing results

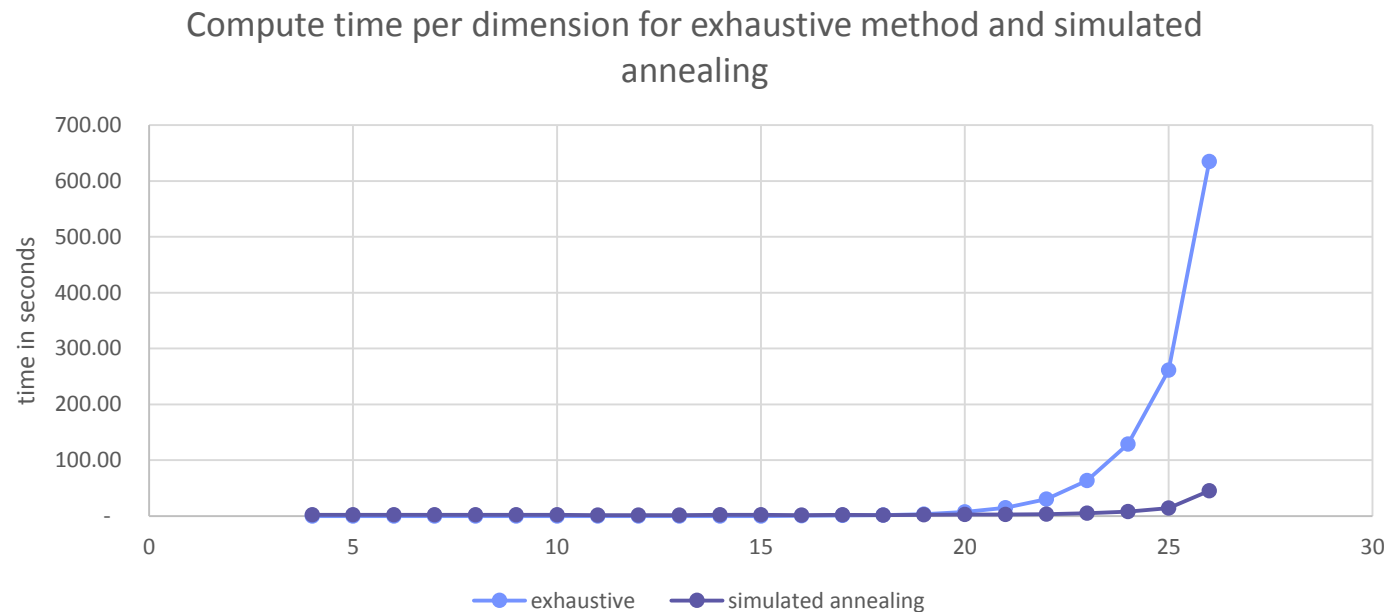
- The distribution of energy levels can not be approximated as shown below:



A simple observation of the distribution of energy levels confirms the difficulty to approximate the negative tail having an isolated scenario at the far end of this tail. This is consistent with the presence of large and tall barriers between minimums.

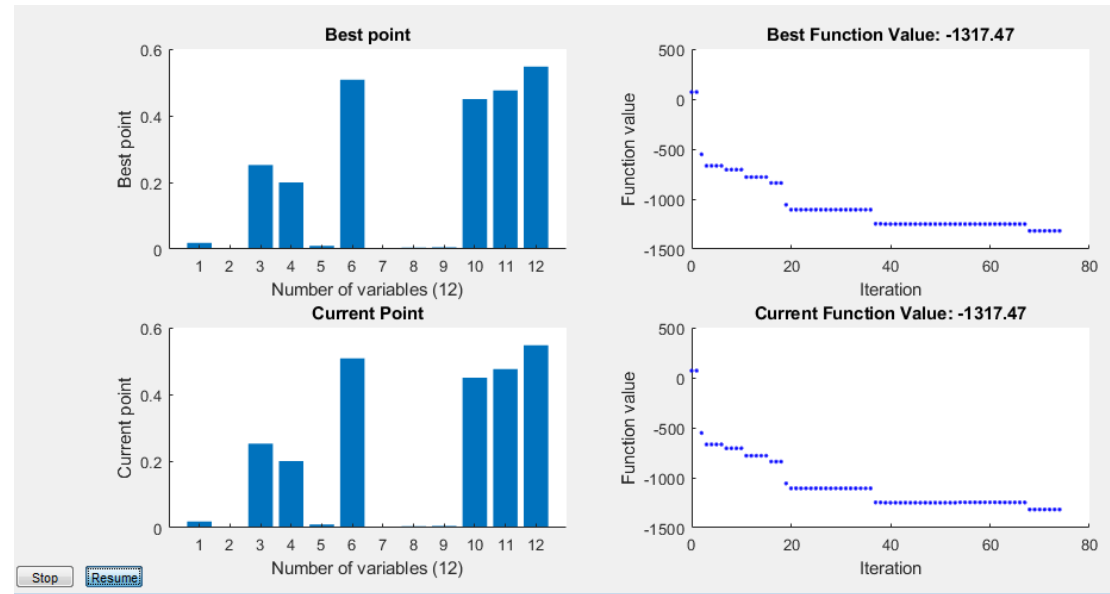
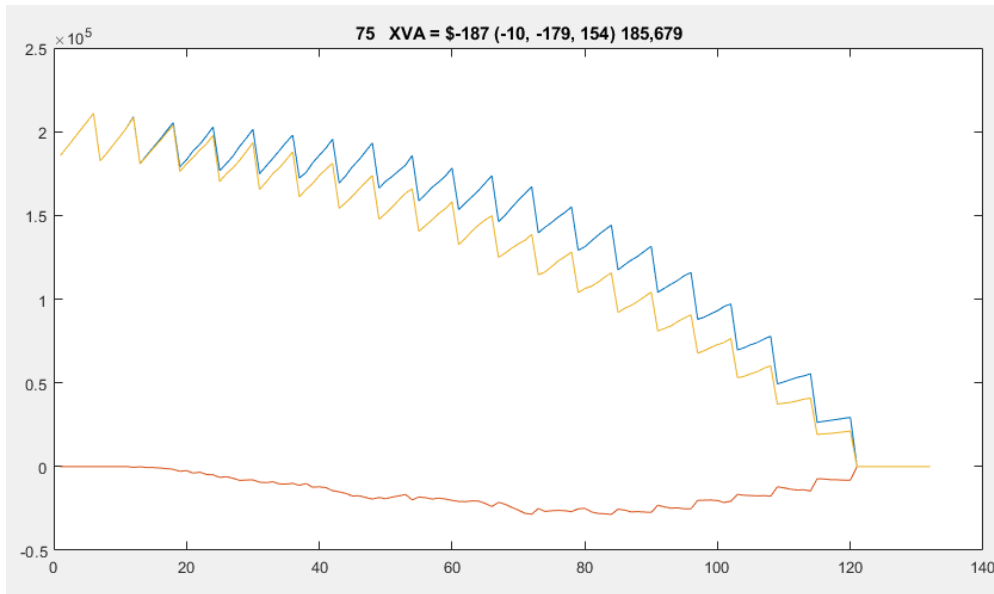
# Optimisation Simulated annealing results

- For another problem involving this time four currencies we have 38 risk factors ( 4 interest rates, 3 foreign exchanges we obtain the following performances using the exhaustive algorithm computing all the combinations of shocks and simulated annealing.
- We therefore slice the vector and the matrix of our QUBO to build smaller equivalent problems from dimension 5 to dimension 20. We obtain the following results:



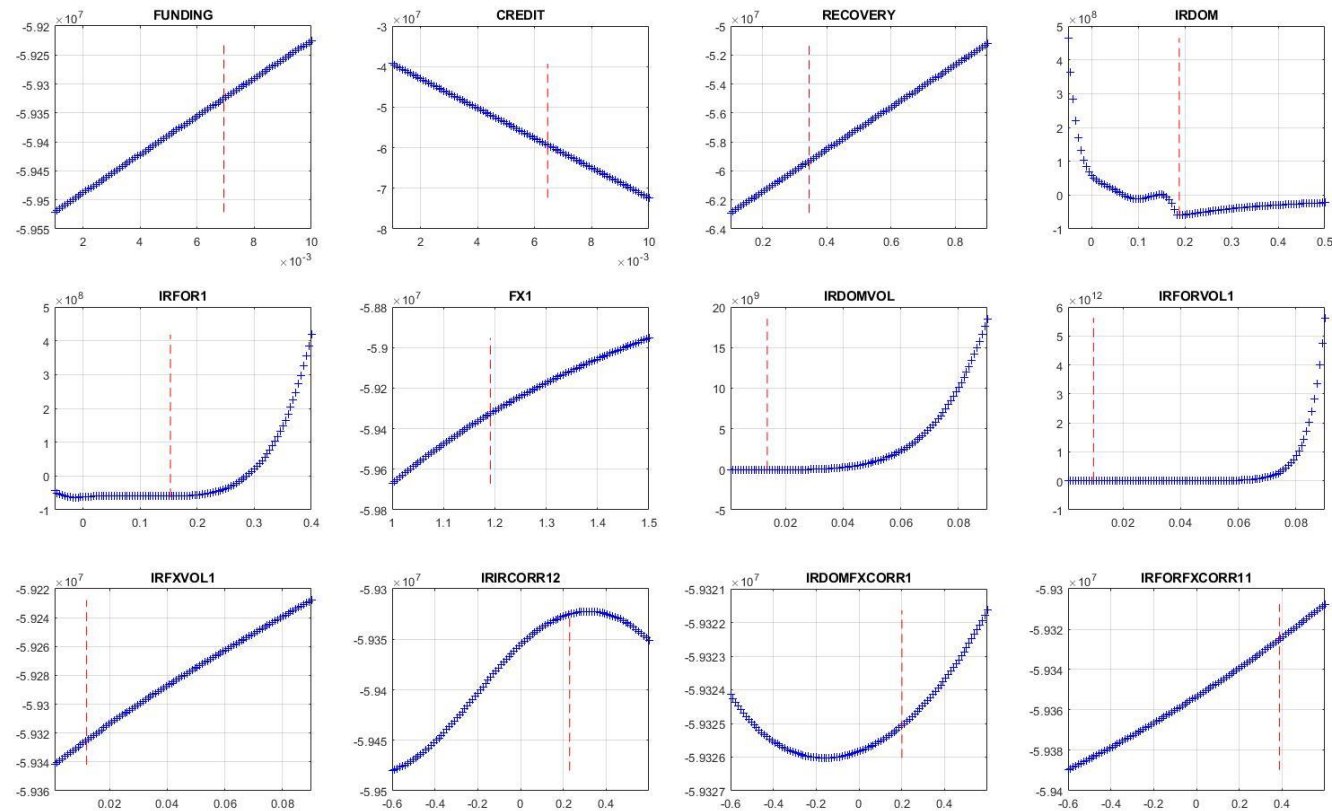
# Optimisation Simulated annealing results

- We can also run the simulated annealing algorithm directly in continuous mode, but:
  - the results are slower to get
  - The solution can't be scaled to the actual shocks of the reverse stress test.



# Optimisation Simulated annealing results

In lower dimension inflexions due to the presence of multiple minimum are difficult to see:



# Optimisation Quantum annealing

- Recent advances in commercial quantum technologies have given rise to widespread industry investigation into their practical usefulness. Quantum computing holds particular promise, as the field aims to tackle the most difficult computational problems known in mathematics, computer science, and physics. Specifically, the quantum processing units (QPUs) produced by D-Wave Systems have been subject to research in a variety of areas, such as the automotive industry, quantum simulations of materials, operations research, and more.
- These QPUs implement a quantum annealing algorithm, where qubits (quantum bits) can be prepared in a simple initial energy configuration where all qubits are both 0 and 1 (quantum superposition), and are then evolved to a final configuration corresponding to a combinatorial optimization problem.
- It has been shown that if this quantum system is evolved carefully enough, the qubits remain in the minimum energy configuration. The qubits can therefore be used to represent variables, whose values (0 or 1) are determined by the QPU in an attempt to minimize the energy of the system.
- The quantum properties of the qubits, such as entanglement, superposition, and tunnelling, are used during the annealing process to compute solutions, and can potentially provide a speed-up over classical algorithms.

# Optimisation Quantum annealing

- The Hamiltonian representing the mapping between the eigen states of the qubits and their energy has two components:
  - The initial Hamiltonian defining the lowest energy state.
  - The final Hamiltonian defining the actual QUBO to solve.
- The quantum annealing process consists in slowly introducing the final Hamiltonian while removing the initial one.
- As the annealing progress, the system is finding closer energy levels from the ground.

$$\mathcal{H}_{ising} = \underbrace{-\frac{A(s)}{2} \left( \sum_i \sigma_x^{(i)} \right)}_{\text{Initial Hamiltonian}} + \underbrace{\frac{B(s)}{2} \left( \sum_i h_i \sigma_z^{(i)} + \sum_{i>j} J_{ij} \sigma_z^{(i)} \sigma_z^{(j)} \right)}_{\text{Final Hamiltonian}} \quad (2.1)$$

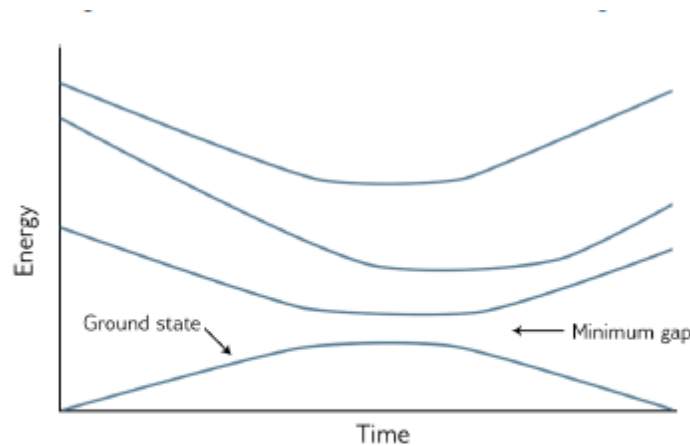


Figure 2.4: Eigenspectrum, where the ground state is at the bottom and the higher excited states are above.

# Optimisation Quantum annealing results

- Interacting with the D-Wave™ system is extremely simple and can be done in python or Matlab by just creating a DWaveSampler, populating with the correct biases (vector) and couplers (matrix), sending a request and receiving the response. The response contains several results and guarantees that the optimum solution is included with a high probability. It is therefore necessary to check which solution is the best.
- Before interacting with the machine, the QUBO needs to be mapped to the topology of the machine. Indeed the working graph representing all the qubits and their connections is not fully connected.

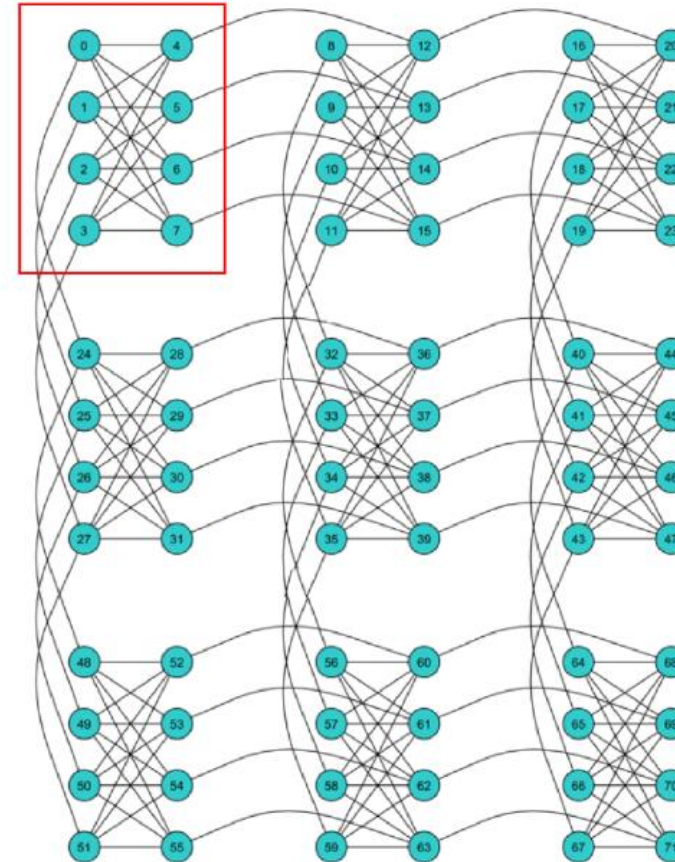


Figure 7: Figure showing the connectivity of the graph from D-Wave™



# Optimisation Quantum annealing results

- The qubits are organised in a Chimera graph. A technique known as "Minor Embedding" is used to merge nodes in the graph to increase the graph connectivity. They need to be "unembedded" when we obtain the result.
- We can therefore use the software provided by D-Wave™ to automatically find a correct embedding. The embedded problem is then solved by the QPU. The problem will have to be unembedded from the output to get the answer to the original problem. For our simple test example the quantum annealing results are as follows:

FUNDING	CREDIT	RECOVERY	IRDOM	IRFOR	FX	IRDOMVOL	IRFORVOL	IRFXVOL	IRIRCORR	IRDOMFXCORR	IRFORFXCORR
0	1	0	0	1	1	1	1	1	1	1	1

Table 5: Table showing the result from QA

- It is the correct solution as it is corresponding to the result from simulated annealing and the exhaustive computation of all combinations.
- Another interesting result is the time spent computing the result (in micro seconds):

1. anneal\_time\_per\_run: 20
2. post\_processing\_overhead\_time: 12208
3. qpu\_access\_overhead\_time: 3046
4. qpu\_access\_time: 23361
5. qpu\_anneal\_time\_per\_sample: 20
6. qpu\_delay\_time\_per\_sample: 21
7. qpu\_programming\_time: 15163
8. qpu\_readout\_time\_per\_sample: 123
9. qpu\_sampling\_time: 8198
10. readout\_time\_per\_run: 123
11. run\_time\_chip: 8198
12. total\_post\_processing\_time: 12208
13. total\_real\_time: 23361

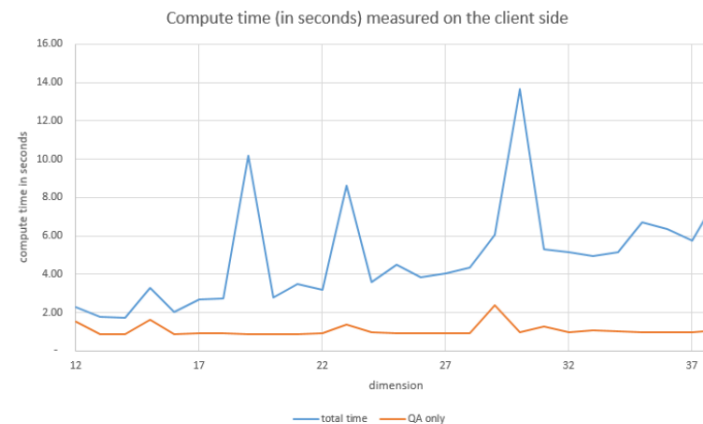
# Optimisation Quantum annealing results

- For this particular example we find total real time really close to the compute time using simulated annealing (0:023361 0:024930).
- This shows that QA is at least as fast as SA. It is important to carry out further tests but with an increasing number of factors to confirm the superiority of QA over.
- A simple way to increase the number of risk factors is to increase the number of currencies and interest rate swaps dependent on them:

Number of currencies	total number of risk factors	value adjustments	interest rates	foreign exchange	correlations
1	5	3	2	0	0
2	12	3	4	2	3
3	19	3	6	4	6
4	38	3	8	6	21

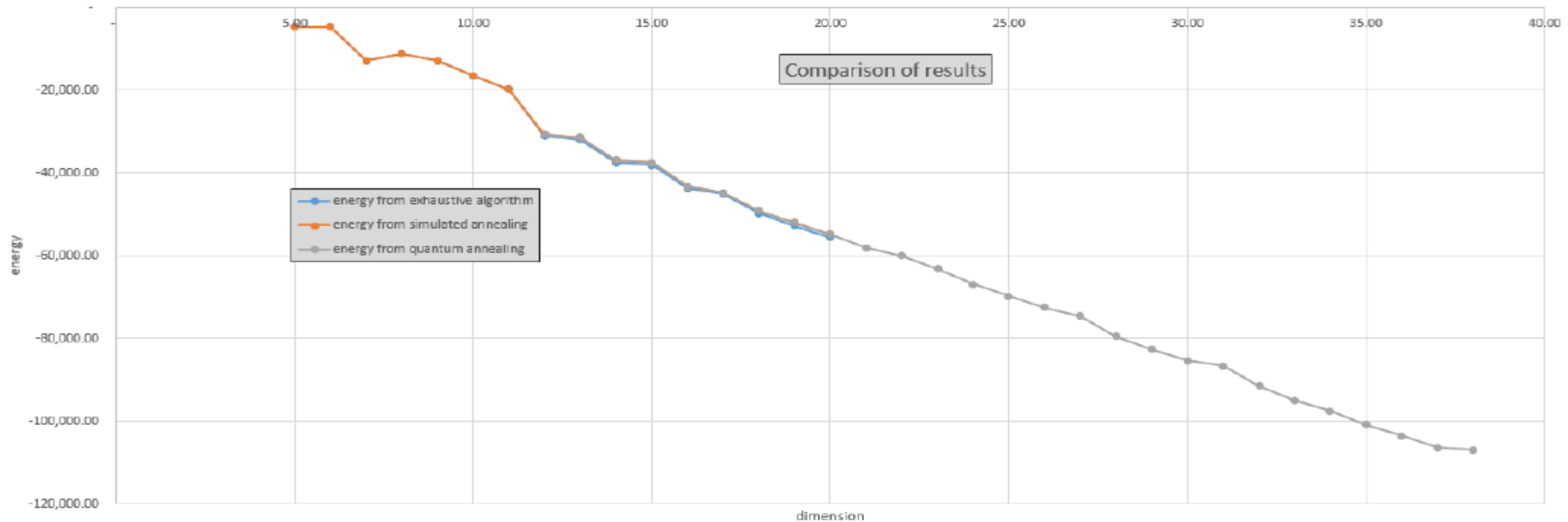
# Optimisation Quantum annealing results

- For a simple convex problem, the QPU time is the same for all dimensions for it is found at the first run.
- For dimension greater than 12, the QPU is faster than the CPU as the compute time on the QPU side.
- If we take into account the time spent to embed and reconstruct the solution, then the compute time increases linearly with the dimension.



# Optimisation Quantum annealing results

Results are in line between CPU and QPU for small dimensions. It is not possible to validate the result for higher dimension but it is reassuring to see the energy decreasing smoothly as the number of dimensions decreases, almost linearly.



# Optimisation Quantum annealing results

- For more complex (less convex) problem we are not reaching the optimum as easily:
  - We get several observations (reads) from the same experiment ( usually 100)
  - It can take several thousands runs to obtain it. We use the time to solution to compare timing between QA and SA, which represents the time per unit run multiplied by the average number of runs necessary to have a high level of confidence.
  - Often post processing involving an optimisation method can be used to speed up the search.
- But already the gain in speed is interesting:
  - Simulate annealing necessitates running the XVA function several thousands times for  $N=12$  input variable. The compute time is increasing exponentially.
  - Building the QUBO necessitates  $4 N$  times the compute time of the XVA function (using AAD, adjoint algorithmic differentiation, adjoint for first derivatives and tangent over adjoint for second derivatives). It is then increasing linearly on the CPU side. If the result can be reached linearly using the QPU then it becomes a really interesting proposition.

=> work in progress

# Conclusion

## Potential problems

- Several potential problems need to be addressed:
  - This methodology is maybe too pessimistic as it relies on the portfolio suffering a loss due to the realisation of the worst combination of factors at each date.
  - The underlying assumption that the worst case scenario over a long period of time is attainable by applying successive worst case scenario can be accepted intuitively but is not proved. It is even possible to find counter examples where for example exercisability can change the direction of a sensitivity. For example, the exercise condition can have short term effect which has an opposite direction to its long term effect.
  - The dynamic aspect of portfolio management is not included. This could be reduced by auto hedging some risk factors with some counterparties.
  - We may need to review the size of shocks as they realise at successive time steps of the simulation. For this we should use the serial correlation of risk factors or a series of realised historical shocks of a particular stress period.

# Conclusion Next

- Financial stress testing is becoming a dominant part of the arsenal built by regulators to protect the economic stability from the eventual distress of one or a series of financial institutions. It is also a useful complement for senior managers and heads of desk who must assess the size of their provisions and manage their tail risk.
- We showed that it is possible to benefit from the speed and power of QA by expressing the XVA reverse stressing as a QUBO (quadratic unconstrained binary optimisation) problem. As of today few attempts to exploit quantum annealing in finance have been successful. One of the reason was the difficulty to find and formulate a problem in a shape that could be solved particularly well with this technique and not or less with others.
- The model has been implemented using D-wave API (<https://www.dwavesys.com/software>) and compared to a simulated annealing benchmark written in Matlab.
- The next stage is to finish the comparison between simulated annealing and quantum annealing results and performances measuring time to solution using production models and real portfolios.
- Already QA gives us a set of sub optimum solutions that can give a valuable insight on the frailties of the portfolio in period of stress.

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